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S. Ishino · **H. Takeuchi** · **M. Tsubota**

Two-superfluid Model of Two-component Bose-Einstein Condensates; First Sound and Second Sound

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Abstract Superfluid ^{4}He at a finite temperature is described by the two-fluid model with the normal fluid component and the superfluid component. We formulate the two-fluid model for two-component BECs, namely two-superfluid model, starting from the coupled Gross-Pitaevskii equations. The two-superfluid model well corresponds to the two-fluid model in superfluid ^{4}He. In a special condition, the two sound modes in the two-superfluid model behave like first and second sounds in the two-fluid model of superfluid ^{4}He.
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1 Introduction

Superfluid ^{4}He has been thoroughly studied theoretically and experimentally in the field of low temperature physics since Kapitsa discovered superfluidity of ^{4}He below the transition temperature[1]. Tisza[2] and Landau[3] succeeded in understanding the superfluidity of ^{4}He with introducing the two-fluid model, which states that the system consists of normal fluid and superfluid being independent of each other and is described by

$$\rho_n\left(\frac{\partial \boldsymbol{v}_n}{\partial t}+(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n\right)=-\frac{\rho_n}{\rho}\nabla P-\rho_n\sigma\nabla T+\eta_n\nabla^2\boldsymbol{v}_n, \tag{1}$$

$$\rho_s\left(\frac{\partial \boldsymbol{v}_s}{\partial t}+(\boldsymbol{v}_s\cdot\nabla)\boldsymbol{v}_s\right)=-\frac{\rho_s}{\rho}\nabla P+\rho_s\sigma\nabla T. \tag{2}$$

Here ρ_n and $\boldsymbol{v}_n$ are density and velocity of normal fluid, and ρ_s and $\boldsymbol{v}_s$ are those of superfluid. σ is entropy per unit mass of the normal fluid, $\rho=\rho_n+\rho_s$ is total

Department of Physics, Osaka City University,Sumiyoshi-Ku, Osaka 558-8585, Japan
Tel.:+81-6-6605-2501
Fax:+81-6-6605-2522
E-mail: ishino@sci.osaka-cu.ac.jp

density and η_n is the coefficient of viscosity of the normal fluid. The pressure gradient ∇P runs both components in the same direction and the thermal gradient ∇T does in the opposite direction. Thermal counterflow driven by a thermal gradient is characteristic of superfluid ^{4}He. When the relative velocity between two components is large, they become dependent through the mutual friction $\boldsymbol{F}_{sn}$, which is added to Eqs. (1) and (2)[4]. Other formulations for superfluid ^{4}He are derived from the conservation law and equations of motion of mass density and entropy density. The hydrodynamic equations[5] are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{j}, \tag{3}$$

$$\frac{\partial \sigma}{\partial t} = -\frac{\rho_s \sigma}{\rho} \nabla \cdot (\boldsymbol{v}_n - \boldsymbol{v}_s), \tag{4}$$

$$\frac{\partial \boldsymbol{j}}{\partial t} = -\nabla P, \tag{5}$$

$$\frac{\partial}{\partial t}(\boldsymbol{v}_n - \boldsymbol{v}_s) = -\frac{\rho \sigma}{\rho_n} \nabla T, \tag{6}$$

where $\boldsymbol{j} = \rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s$. These equations yield the wave equation of mass density and entropy density, which leads to first sound and second sound. First sound is a mode of oscillation of total density and it exists generally in a usual fluid. While, second sound is a characteristic mode in superfluid ^{4}He, in which entropy oscillates without oscillating total density, not existing in a usual fluid.

An atomic BEC is one of the most important subjects in modern physics. Especially, two-component BECs are known to create various exotic structure of quantized vortices[6] and cause some characteristic hydrodynamic instability such as Kelvin-Helmholtz instability[7] and Rayleigh-Taylor instability[8]. In another paper, we investigate counterflow in two-component BECs which has many analogies with thermal counterflow in superfluid ^{4}He. For example, when the relative velocity exceeds a critical value, the counterflow becomes unstable and quantum turbulence appears like in thermal counterflow. In this work, we describe two-component BECs following the two-fluid model of superfluid ^{4}He and obtain four equations similar to Eqs. (3)-(6). We derive two sound modes from these elementary equations. Two sound modes in two-component BECs were obtained by some other works[9], but we have them correspond to first and second sounds. This is the main point of this work. Thus we can expect to improve interactive studies in superfluid ^{4}He and two-component BECs with investigating common features between these.

2 The two-fluid model in two-component BECs

We consider binary mixture of BECs described by the wave functions $\Psi_j = \sqrt{n_j} e^{i\phi_j}$ in the mean-field approximation at $T = 0$ K, where the index j refers to each component j ($j = 1, 2$). The wave functions Ψ_j are governed by the coupled Gross-Pitaevskii (GP) equations[10],

$$i\hbar \frac{\partial}{\partial t} \Psi_1 = -\frac{\hbar^2}{2m_1} \nabla^2 \Psi_1 + V(\boldsymbol{r}) \Psi_1 + g_{11} |\Psi_1|^2 \Psi_1 + g_{12} |\Psi_2|^2 \Psi_1, \tag{7}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2 = -\frac{\hbar^2}{2m_2}\nabla^2\Psi_2 + V(\boldsymbol{r})\Psi_2 + g_{22}|\Psi_2|^2\Psi_2 + g_{12}|\Psi_1|^2\Psi_2, \tag{8}$$

where m_j is particle mass associated with the species, g_{jj} is intracomponent interaction and g_{12} is intercomponent interaction. We insert Ψ_j into Eqs. (7) and (8) and obtain the hydrodynamic equations,

$$\frac{\partial\rho_1}{\partial t} = -\nabla\cdot(\rho_1\boldsymbol{v}_1), \tag{9}$$

$$\frac{\partial\rho_2}{\partial t} = -\nabla\cdot(\rho_2\boldsymbol{v}_2), \tag{10}$$

$$\begin{aligned}\rho_1\frac{\partial}{\partial t}\boldsymbol{v}_1 &= \rho_1\nabla\Big\{\frac{\hbar^2}{2m_1^2\sqrt{\rho_1}}\triangle\sqrt{\rho_1} - \frac{1}{2}v_1^2 - \frac{1}{m_1^2}V\Big\}\\ &\quad -\rho_1\nabla\Big(\frac{g_{11}}{m_1^2}\rho_1 - \frac{g_{12}}{m_1 m_2}\rho_2\Big),\end{aligned} \tag{11}$$

$$\begin{aligned}\rho_2\frac{\partial}{\partial t}\boldsymbol{v}_2 &= \rho_2\nabla\Big\{\frac{\hbar^2}{2m_2^2\sqrt{\rho_2}}\triangle\sqrt{\rho_2} - \frac{1}{2}v_2^2 - \frac{1}{m_2^2}V\Big\}\\ &\quad -\rho_2\nabla\Big(\frac{g_{22}}{m_2^2}\rho_2 - \frac{g_{12}}{m_1 m_2}\rho_1\Big).\end{aligned} \tag{12}$$

where $\rho_j = m_j n_j$ is mass density and $\boldsymbol{v}_j = \frac{\hbar}{m_j}\nabla\phi_j$ is superfluid velocity. Equations (9) and (10) are equations of continuity for ρ_j and Eqs. (11) and (12) are quasi-Euler equations for the superfluid velocity.

We will derive equations similar to Eqs. (1) and (2) to reveal correspondence between superfluid ^{4}He and two-component BECs. Here we consider a uniform system and apply the long-wavelength approximation, so the potential term and the quantum pressure term in Eqs. (11) and (12) are neglected. The pressure of the whole system is

$$\tilde{P} = \frac{g_{11}\rho_1^2}{2m_1^2} + \frac{g_{22}\rho_2^2}{2m_2^2} + \frac{g_{12}\rho_1\rho_2}{m_1 m_2},$$

since $P = -\partial E/\partial V$. Then Eqs. (11) and (12) turn into

$$\rho_1\Big(\frac{\partial\boldsymbol{v}_1}{\partial t} + (\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v}_1\Big) = -\frac{1}{2}\nabla\tilde{P} - \frac{1}{2}\tilde{\nabla}\tilde{T}, \tag{13}$$

$$\rho_2\Big(\frac{\partial\boldsymbol{v}_2}{\partial t} + (\boldsymbol{v}_2\cdot\nabla)\boldsymbol{v}_2\Big) = -\frac{1}{2}\nabla\tilde{P} + \frac{1}{2}\tilde{\nabla}\tilde{T}, \tag{14}$$

with

$$\tilde{\nabla}\tilde{T} = \frac{g_{11}}{2m_1^2}\nabla\rho_1^2 - \frac{g_{22}}{2m_2^2}\nabla\rho_2^2 + \frac{g_{12}}{m_1 m_2}(\rho_1\nabla\rho_2 - \rho_2\nabla\rho_1).$$

It is impossible to describe the right hand side by gradient of some scalar potential because ρ_1 and ρ_2 are spatially dependent, but we represent it by $\tilde{\nabla}\tilde{T}$ in order to emphasize the correspondence to ∇T in Eqs. (1) and (2). From Eqs. (13) and (14), we can find that two-component BECs are driven by two terms. The pressure gradient $\nabla\tilde{P}$ runs both components in the same direction, while $\tilde{\nabla}\tilde{T}$ runs them oppositely. This nature is just the same as one of the two-fluid model in superfluid ^{4}He.

3 First sound and second sound in two-component BECs

In this section, we will derive two sound modes from the four elementary equations in two-component BECs and let them correspond to first and second sounds. Here we assume that superfluid velocities $\boldsymbol{v}_j$ are small and the non-linear terms are neglected. By making Eq. (9) $\pm$ Eq. (10) and Eq. (11) $\pm$ Eq. (12) we obtain

$$\frac{\partial}{\partial t}\rho_+ = -\nabla\cdot\boldsymbol{j}_+, \tag{15}$$

$$\frac{\partial}{\partial t}\rho_- = -\nabla\cdot\boldsymbol{j}_-, \tag{16}$$

$$\frac{\partial}{\partial t}\boldsymbol{j}_+ = -\nabla\left\{\frac{g_{11}}{8m_1^2}(\rho_+ + \rho_-)^2 + \frac{g_{22}}{8m_2^2}(\rho_+ - \rho_-)^2 + \frac{g_{12}}{4m_1m_2}(\rho_+^2 - \rho_-^2)\right\}, \tag{17}$$

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{j}_- = &-\nabla\left\{\frac{g_{11}}{8m_1^2}(\rho_+ + \rho_-)^2 - \frac{g_{22}}{8m_2^2}(\rho_+ - \rho_-)^2\right\}\\ &+\frac{g_{12}}{4m_1m_2}\left\{(\rho_+ + \rho_-)\nabla(\rho_+ - \rho_-) - (\rho_+ - \rho_-)\nabla(\rho_+ + \rho_-)\right\},\end{aligned} \tag{18}$$

where $\rho_\pm \equiv \rho_1 \pm \rho_2$ and $\boldsymbol{j}_\pm \equiv \rho_1\boldsymbol{v}_1 \pm \rho_2\boldsymbol{v}_2$. Because first sound means oscillation of ρ with two components in phase and second sound means oscillation of σ with them out of phase, we expect that oscillations of ρ_+ and ρ_- correspond respectively to first and second sounds. Now the right hand sides of Eqs. (17) and (18) should be $\nabla\tilde{P}$ and $\tilde{\nabla}\tilde{T}$ respectively. We can write $\nabla\tilde{P}$ and $\tilde{\nabla}\tilde{T}$ as functional of ρ_+ and ρ_- by

$$\nabla\tilde{P} = \mathrm{A}\nabla\rho_+ + \mathrm{B}\nabla\rho_-, \tag{19}$$

$$\tilde{\nabla}\tilde{T} = \mathrm{C}\nabla\rho_+ + \mathrm{D}\nabla\rho_-, \tag{20}$$

where

$$\begin{aligned}\mathrm{A} &= \frac{g_{11}}{4m_1^2}(\rho_+ + \rho_-) + \frac{g_{22}}{4m_2^2}(\rho_+ - \rho_-) + \frac{g_{12}}{2m_1m_2}\rho_+,\\ \mathrm{B} &= \frac{g_{11}}{4m_1^2}(\rho_+ + \rho_-) - \frac{g_{22}}{4m_2^2}(\rho_+ - \rho_-) - \frac{g_{12}}{2m_1m_2}\rho_-,\\ \mathrm{C} &= \frac{g_{11}}{4m_1^2}(\rho_+ + \rho_-) - \frac{g_{22}}{4m_2^2}(\rho_+ - \rho_-) + \frac{g_{12}}{2m_1m_2}\rho_-,\\ \mathrm{D} &= \frac{g_{11}}{4m_1^2}(\rho_+ + \rho_-) + \frac{g_{22}}{4m_2^2}(\rho_+ - \rho_-) - \frac{g_{12}}{2m_1m_2}\rho_+.\end{aligned}$$

The wave equations derived from Eq.(15)-(18) are reduced to

$$\frac{\partial^2}{\partial t^2}\rho_+ = \mathrm{A}\nabla^2\rho_+ + \mathrm{B}\nabla^2\rho_-, \tag{21}$$

$$\frac{\partial^2}{\partial t^2}\rho_- = \mathrm{C}\nabla^2\rho_+ + \mathrm{D}\nabla^2\rho_-. \tag{22}$$

Considering the plane waves that ρ_+ and ρ_- oscillate around the equilibrium values ρ_+^0 and ρ_-^0 with the frequency ω and the wave number $\boldsymbol{k}$ like

$$\begin{aligned}\rho_+ &= \rho_+^0 + \delta\rho_+ \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)],\\ \rho_- &= \rho_-^0 + \delta\rho_- \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)],\end{aligned}$$

the sound velocities are

$$\begin{aligned}c^2 = &\frac{g_{11}}{4m_1^2}(\rho_+^0 + \rho_-^0) + \frac{g_{22}}{4m_2^2}(\rho_+^0 - \rho_-^0)\\ &\pm\sqrt{\left\{\frac{g_{11}}{4m_1^2}(\rho_+^0 + \rho_-^0) - \frac{g_{22}}{4m_2^2}(\rho_+^0 - \rho_-^0)\right\}^2 + \frac{g_{12}^2}{4m_1^2 m_2^2}(\rho_+^{0^2} - \rho_-^{0^2})},\end{aligned} \quad (23)$$

where $c \equiv \omega/|\boldsymbol{k}|$. These are obtained from the dispersion relation of the Bogoliubov excitations[9,10] in two-component BECs in the limit of long wavelength.

In superfluid ^{4}He first and second sounds are modes that ρ and σ independently oscillate. However, Eq. (23) does not necessarily describe first and second sounds because two modes are mixed. We can find that ρ_+ and ρ_- oscillate independently when B and C vanish in Eqs. (21) and (22). This conditions are reduced to

$$\rho_1^0 = \rho_2^0, \qquad \frac{g_{11}}{m_1^2} = \frac{g_{22}}{m_2^2}.$$

Then sound velocities of the two modes are

$$c_\pm^2 = s^2 \pm \frac{g_{12}\rho^0}{m_1 m_2}, \quad (24)$$

where $s = \sqrt{g_{11}\rho_1^0/m_1^2} = \sqrt{g_{22}\rho_2^0/m_2^2}$ and $\rho^0 = \rho_1^0 = \rho_2^0$. The mode of c_+^2 is oscillation of ρ_+, first sound, and the mode of c_-^2 is oscillation of ρ_-, second sound. First sound velocity increases with g_{12} and second sound velocity decreases with g_{12} (Fig.1). When $|g_{12}| > g \equiv \sqrt{g_{11}g_{22}}$, c_+ or c_- becomes imaginary so that the dynamical instability leads to the collapse or the phase separation in the two-component BECs[10].

4 Summary

We formulated the two-fluid model for two-component BECs, starting from the coupled GP equations. This model well corresponds to the two-fluid model in superfluid ^{4}He expect for the mutual friction term. We obtained the condition that two sound modes are independent of each other like first and second sounds in superfluid ^{4}He. Second sound has an important role to investigate quantum turbulence in superfluid ^{4}He. We are interested in how second sound interacts with a vortex in two-component BECs. In the future, we should investigate ”mutual friction” induced by the interaction between vortices and the Bogoliubov excitations in two-component BECs. The details will be reported soon elsewhere.

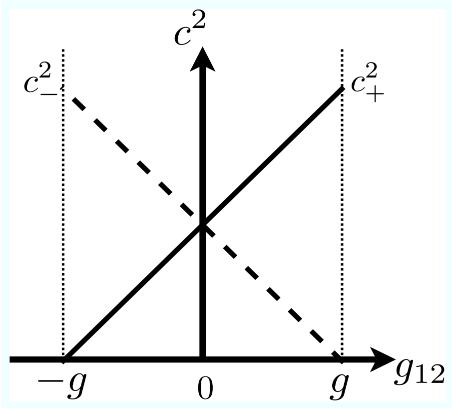

Fig. 1 Velocity of first and second sounds as a function of g_{12}. The solid and dashed line refers to first sound c_+^2 and second sound c_-^2 respectively.

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References

1. P. L. Kapitza, Nature **141**, 74 (1937).
2. L. Tisza, Nature **141**, 913 (1938).
3. L. Landau, J. Phys. U.S.S.R. **5**, 71 (1941).
4. C. J. Gorter and J. H. Melink, Physica. **15**, 285 (1949).
5. I. M. Khalatnikov, An Introduction to the Theory of Superfluidity (Perseus Publishing, 1965).
6. K. Kasamatsu, M. Tsubota, and M. Ueda, Int.J. Mod. Phys. B **19**, 1835 (2005).
7. H. Takeuchi, N. Suzuki, K. Kasamatsu, H. Saito, and M. Tsubota, Phys. Rev. B **81**, 094517 (2010).
8. K. Sasaki, N. Suzuki, D. Akamatsu, and H. Saito, Phys. Rev. A **80**, 063611 (2009).
9. Yu. G. Gladush, A. M. Kamchatnov, Z. Shi, P. G. Kevrekidis, D. J. Frantzeskakis, and B. A. Malomed, Phys. Rev. A **79**, 033623 (2009).
10. C. J. Pethick and H. Smith, BOSE-EINSTEIN CONDENSATION IN DILUTE GASES Second Edition (Cambridge University Press, Cambridge, 2008).